EPJ manuscript No.
(will be inserted by the editor)

Synchronization of multi-phase oscillators: An Axelrod-inspired model

Marcelo N. Kuperman and Damián H. Zanette

Consejo Nacional de Investigaciones Científicas y Técnicas, Centro Atómico Bariloche and Instituto Balseiro, 8400 Bariloche, Río Negro, Argentina

Received: date / Revised version: date

Abstract. Inspired by Axelrod's model of culture dissemination, we introduce and analyze a model for a population of coupled oscillators where different levels of synchronization can be assimilated to different degrees of cultural organization. The state of each oscillator is represented by a set of phases, and the interaction –which occurs between homologous phases– is weighted by a decreasing function of the distance between individual states. Both ordered arrays and random networks are considered. We find that the transition between synchronization and incoherent behaviour is mediated by a clustering regime with rich organizational structure, where some of the phases of a given oscillator can be synchronized to a certain cluster, while its other phases are synchronized to different clusters.

PACS. 05.45.Xt Synchronization; coupled oscillators – 87.23.Ge Dynamics of social systems – 89.75.Fb Structures and organization in complex systems

1 Introduction

Robert Axelrod's model for the dissemination of culture [1] aims at giving a simplified picture of the processes that shape the distribution of cultural features in a society, in particular, the balance between the trend to convergence due to cultural affinity, and the mechanisms that maintain diversity. This model has attracted the attention of physicists because of its nontrivial phenomenology, which includes the existence of a variety of absorbing states, critical phenomena, and non-monotonic dynamics [2]. Among other topics, the transition between homogeneous and heterogeneous from states under variation of the number of cultural traits [3], the effects of noise [4] and non-trivial underlying topologies [5], as well as the mean-field limit [6] have been characterized.

In its original formulation [1], Axelrod's model described the cultural profile of each agent in a population by a vector of *features*. Each feature can adopt a number of *traits*. Agents are situated at the nodes of a bidimensional square array, and interaction occurs between nearest neighbours. Two neighbour agents interact with a probability proportional to their cultural overlap, i.e. to number of features in their profiles whose traits coincide. As the result of the interaction, the profiles change, approaching each other. If the two profiles exhibit completely different traits, interaction is impossible. This system reaches a frozen state where any two neighbours have either identical or completely different profiles. The final state can be homogeneous, with all the population having the same profile, or consist of coexisting domains, thus maintaining cultural diversity [1,2]. It turns out that, as the number of possible traits per feature grows, there is a transition from homogeneity to diversity [3].

From a physicist's viewpoint, there is a strong similarity between the self-organization process that determines the distribution of cultural profiles in Axelrod's model, and the evolution of a spatially extended system able to develop either homogeneous states or phase coexistence. Synchronization of distributed interacting oscillators is an example [7,8,9]. In these systems, the attractive interactions that may lead to the formation of synchronized domains plays a role similar to the tendency of interacting individuals to become culturally more similar in Axelrod's model. On the other hand, diversity between individual attributes of oscillators impedes their synchronization, like too much cultural divergence does not allow interaction.

Inspired by Axelrod's model, in this paper we introduce and analyze a system of interacting oscillators, where the state of each oscillator is characterized by a set of several phases. Each phase would correspond, in Axelrod's model, to a cultural feature. Homologous phases of different oscillators are coupled to each other, with a coupling intensity that depends on the overall distance between the states of the two oscillators. We show that, in this system, the synchronization transition can be straightforwardly identified with Axelrod's transition from cultural homogeneity to diversity. Additionally, we disclose an intermediate regime with rich organizational structure, where clusters of mutually synchronized oscillators, different for each phase, spontaneously appear.

2 Model

We consider a population of N oscillators, each of them occupying a node in a network. The state of oscillator i is characterized by a set of F phases, $\phi_i^f(t) \in (0, 2\pi)$ ($f = 1, \dots, F$). The dynamics of the phases $\phi_i^f(t)$ is governed by Kuramoto-like equations [10,9]

$$\dot{\phi}_i^f = \frac{1}{\nu_i} \sum_{j \in \mathcal{N}_i} k_{ij} \sin(\phi_j^f - \phi_i^f), \tag{1}$$

where the sum runs over the oscillators j connected to i by network links, which define the neighbourhood $\mathcal{N}_i$ of i. The number of oscillators in $\mathcal{N}_i$ is ν_i. The non-negative coupling constant k_{ij} is computed as a prescribed function of the distance D_{ij} between i and j,

$$k_{ij} = K(D_{ij}). \tag{2}$$

The distance D_{ij}, in turn, characterizes the difference between the individual states of oscillators i and j, as

$$D_{ij} = \frac{1}{2F} \sum_{f=1}^{F} \left[1 - \cos(\phi_j^f - \phi_i^f)\right]. \tag{3}$$

Note that D_{ij} is non-negative, and $D_{ij} = 0$ if and only if $\phi_j^f = \phi_i^f$ for all $f = 1, \dots, F$. Moreover, the maximal possible value of the distance is $D_{ij} = 1$. For the function $K(D)$, which defines the coupling constant through Eq. (2), we choose

$$K(D) = \begin{cases} 1 - (\alpha D)^r & \text{for } D < \alpha^{-1}, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

with $\alpha > 1$ and $r > 0$. Thus, the coupling constant is maximal, $k_{ij} = 1$, when the distance D_{ij} vanishes, and decreases monotonically as D_{ij} grows. It reaches zero for $D_{ij} = \alpha^{-1}$, and $k_{ij} = 0$ for larger distances. The exponent r controls the behaviour of k_{ij} at small distances: for $r < 1$ and $r > 1$, the coupling constant at $D_{ij} = 0$ displays, respectively, a cusp and a flat maximum. For $r = 1$, the dependence of k_{ij} on the distance is linear.

When the coupling constant k_{ij} is positive, the summand in the right-hand side of Eq. (1) represents an attractive interaction between ϕ_i^f and ϕ_j^f. Under its action, the two phases will tend to become mutually synchronized. While the functional dependence of the interaction couples phases separately for each value of $f = 1, \dots, F$, its intensity –determined by the coupling constant– is given by the joint contribution of the difference of all the F phases between the two oscillators through their distance D_{ij}, Eq. (3).

The multi-phase oscillator system defined by Eqs. (1) to (4) can be qualitatively compared to Axelrod's model as follows. The F phases of each oscillator are associated with an agent's cultural features. Different values of each phase –which, in our system, vary continuously over $(0, 2\pi)$– correspond to the traits of Axelrod's model. The distance D_{ij} gives a measure of the cultural dissimilarity between two agents. If it is too large, the two agents do not interact. For small distances, on the other hand, the intensity –or, in Axelrod's model, the probability– of the interaction grows as the distance decreases. Synchronization between phases of two or more oscillators can be assimilated to their cultural consensus. This may occur with respect to all the phases –i.e., all the cultural features– or just some of them. A given oscillator could have some of the phases synchronized with a certain part of the population, and the other phases synchronized with another part, thus giving rise to a state of partial synchronization equivalent to a rich diversity of cultural domains. As in other variants of the model [2], we are here considering that the structure of the population is defined by a generic network, instead of the bidimensional array originally considered by Axelrod.

In our system, we expect that collective synchronization is easier to achieve when the interaction range with respect to the distance D_{ij} between the oscillators' states, given by α^{-1} in Eq. (4), is larger. This corresponds, in Axelrod's picture, to a small number of traits per cultural feature. It has been shown, in fact, that Axelrod's model exhibits a transition between cultural homogeneity and diversity as the number of traits in each feature grows [3, 2]. Similarly, in Eq. (1) a synchronization transition is expected to occur depending on the interaction range α^{-1} being larger or smaller than the typical distance between the states of any two oscillators in the population.

Assume that the system is in a fully unsynchronized state, with all the F phases of all oscillators uniformly distributed over $(0, 2\pi)$. The expected average value for the distance D_{ij} between any two oscillators is

$$\begin{aligned} \langle D_{ij} \rangle &= \frac{1}{8\pi^2 F} \sum_{f=1}^{F} \int_0^{2\pi} d\phi_i^f \int_0^{2\pi} d\phi_j^f \left[1 - \cos(\phi_j^f - \phi_i^f)\right] \\ &= \frac{1}{2}. \end{aligned} \tag{5}$$

If this mean distance is larger than α^{-1}, the unsynchronized state should be stable as oscillator pairs do not interact on the average. Consequently, we predict a transition from synchronization to desynchronization at the critical point

$$\alpha_c = 2. \tag{6}$$

The width of this transition should be controlled by the dispersion in the values of D_{ij}. Under the same assumptions as for the calculation of $\langle D_{ij} \rangle$, the mean square dispersion of the distance turns out to be

$$\sigma_D = \left[\langle D_{ij}^2 \rangle - \langle D_{ij} \rangle^2\right]^{1/2} = \frac{1}{\sqrt{8F}}. \tag{7}$$

According to this estimation, thus, the transition width should decrease as the number of phases F grows. On the other hand, since our estimation is based on the evaluation of the average distance between just two oscillators, the width does not depend on the population size N.

In the following section, we present numerical results for our model, Eqs. (1) to (4), for populations distributed

over ordered and random networks. We confirm the above predictions on the synchronization transition, and show that around the critical point the population segregates into clusters of mutually synchronized oscillators, displaying a high degree of organizational diversity.

3 Numerical results

The results presented in this section are the outcome of numerical calculations made on populations of various sizes, ranging from $N = 100$ to 1000 oscillators. As for the underlying interaction network, we consider a one-dimensional ordered array and random Erdős-Rényi networks [11]. In the former, each node is connected to its four nearest neighbours, two to each side. In the latter, the average number of neighbours per node equals four.

Bearing in mind the analogy with Axelrod's model, we focus our attention on the organization of the population into synchronized clusters at asymptotically long times. A synchronized cluster is a group of oscillators whose phases coincide. Given that we deal with multi-phase oscillators, where the state of each of them is described by F phases, we analyze the organization into groups for each phase separately. Numerically, we consider that two phases are synchronized if they differ in less than $\Delta\phi = 10^{-4}$. To statistically characterize the organization into groups we measure the normalized mean number of groups g and the dispersion σ_g, averaging over phases and realizations. The normalization of the number of groups is performed with respect to its maximum value, N.

A more detailed characterization of the organization into groups is achieved by introduced a normalized Hamming distance, h, averaged over phases and oscillator pairs, as follows. First, we represent the synchronization state of each oscillator pair by defining a set of F matrices M^f ($f = 1, \dots, F$), $N \times N$ in size, with elements

$$M_{ij}^f = \begin{cases} 1 & \text{if } \phi_i^f \text{ and } \phi_j^f \text{ are synchronized,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The Hamming distance between two of these matrices, M^f and $M^{f'}$, is defined as

$$H_{ff'} = \sum_{i,j=1}^{N} |M_{ij}^f - M_{ij}^{f'}|. \tag{9}$$

It equals zero if and only if the two matrices are identical, and its maximum value is $N(N-1)$. The normalized Hamming distance h is obtained by averaging $H_{ff'}$ over all the matrix pairs and normalizing to the maximum value.

Figure 1 shows results for the normalized number of groups g as a function of the parameter α of Eq. (4), for various values of the exponent r and both network topologies. A feature common to all curves is the rather sharp transition from small to large values of g as α grows, just above the critical value predicted in the preceding section,

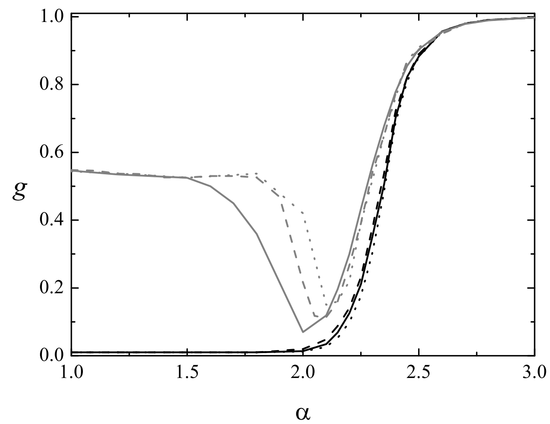

Fig. 1. Normalized number of groups as a function of α for the one-dimensional array (grey) and for random networks (black) with $N = 100$ and $F = 50$. Each set of curves corresponds to three values of the exponent $r = 0.1$ (solid), 1 (dashed), and 3 (dotted).

Eq. (6). This corresponds, as expected, to a desynchronization transition as the range of coupling decreases. To the right of the transition, in fact, the number of groups equals the population size, reveling a state where all oscillators are mutually unsynchronized.

Around and above the transition the behaviour is rather independent of the parameters and of the underlying topology, but a substantial difference is apparent for $\alpha < 2$. While in random networks the normalized number of groups remains small and close to its minimal value $g = 1/N$, revealing full synchronization of all oscillators in all their phases, for the one-dimensional array g attains rather large values, $g \gtrsim 0.5$. By inspecting single realizations on the one-dimensional array, we have verified that these large values of g are due to the occasional occurrence of the so-called *twisted states* [12]. In twisted states, at asymptotically long times, oscillator phases are not synchronized but vary linearly along the array. These spatially-correlated distributions of phases are possible due to the underlying topological order. From the viewpoint of our approach, however, they represent fully unsynchronized states where the number of groups equals the size of the population, N. Therefore, their contribution makes the average normalized number of groups g grow.

For larger values of α, as illustrated by Fig.1, differences between results for random networks and the one-dimensional array, as well as for different values of r, are much less pronounced. Thus, in our analysis for $\alpha > 2$, we focus attention on random networks and fix $r = 1$.

We first verify the prediction made in the preceding section, Eq. (7), that the width of the transition should decrease with as the number of phases F grows. This is confirmed by the numerical results shown in Fig. 2, where we plot the normalized number of groups g as a function of α for three values of F. Figure 3 shows that, in agreement

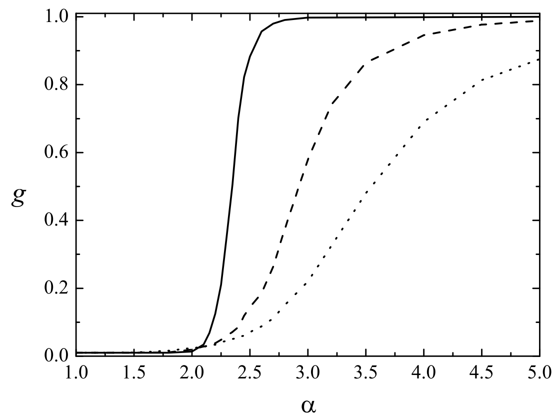

Fig. 2. Normalized number of groups g as a function of α for $N = 100$ and $F = 50$ (solid), 10 (dashed), and 5 (dotted).

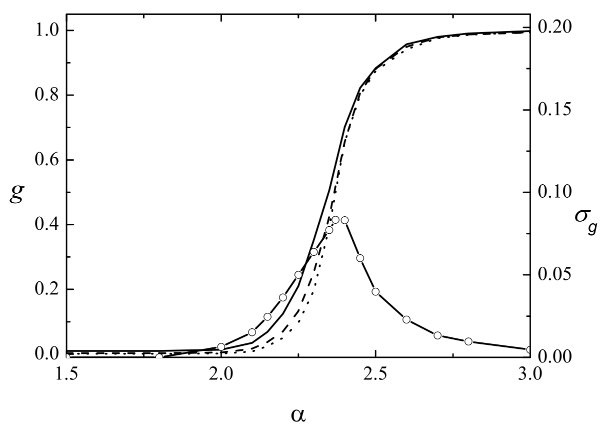

Fig. 3. Normalized number of groups g as a function of α for $F = 50$ and $N = 100$ (solid), $N = 300$ (dashed), and $N = 1000$ (dotted). Empty dots represent the mean square dispersion σ_g of the normalized number of groups for $N = 100$.

with our prediction, there is practically no dependence of the transition width on the population size N.

In contrast with the situation for small α, the growth of g in the zone of the transition, $\alpha \gtrsim 2$, is due to the disaggregation of the fully synchronized state into clusters of mutually synchronized oscillators. Clusters may have various sizes and, consequently, the number of clusters may vary between realizations. Also, as we discuss later, oscillators which are mutually synchronized in some of their phases need not be synchronized in all of them, so that the structure of clustering in a given population is not necessarily the same for each phase. We call this mixed clustering structure *cross synchronization*. The phenomenon of cross synchronization reveals a very rich form of self-organization, also by comparison with the multi-cultural configurations of Axelrod's model.

Empty dots in Fig. 3 represent the mean square dispersion, over phases and realizations, of the normalized number of groups, for $N = 100$ and $F = 50$. We recognize the typical fluctuation peak around the critical point, indicating –in our case– that the number of groups is less well-defined in the transition range where synchronization breaks down. A more detailed picture of the statistics of groups at the transition is provided by Fig.4, where we show histograms of the individual values of the normalized number of groups for several values of α. As α grows, the histogram shifts from a sharp peak at $g \approx 0$ to a sharp peak at $g \approx 1$, passing by broader distributions of variable width as the transition proceeds.

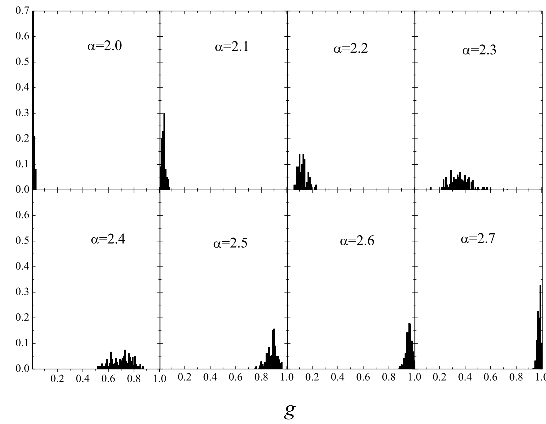

Fig. 4. Histograms of the normalized number of groups g for several values of α, corresponding to the realizations for $N = 100$ used to calculate the averages and mean square dispersions shown in Fig. 3
.

The phenomenon of cross synchronization addressed to above –i.e., the formation of different clustering structures for different phases– can be quantified using the normalized Hamming distance h. In fact, if the distribution of oscillators into clusters is the same in all phases, the matrices M_{ij}^f ($f = 1, \dots, F$) are all identical, and the normalized Hamming distance vanishes. On the other hand, different clustering structures for different phases generally give a positive value of h. For a quantitative evaluation of cross synchronization, the value of h must be compared with the normalized Hamming distance h_r for a random distribution of phases. Numerically, h_r can be computed by randomly shuffling the phases of all oscillators. Figure 5 shows, as full dots, the ratio h/h_r as a function of α for $N = 100$ and $F = 50$ on a random network. Below the transition, as expected, $h/h_r = 0$. In fact, in the fully synchronized state only one group –containing all the population– is formed in all phases. For $\alpha \gg 2$, on the other hand, $h/h_r \approx 1$, showing that well above the transition the unsynchronized state is statistically equivalent to a random distribution of phases. In the intermediate zone, in contrast, h/h_r exhibits a sharp maximum attaining values close to 20. It is therefore in this region that cross synchronization is most conspicuous.

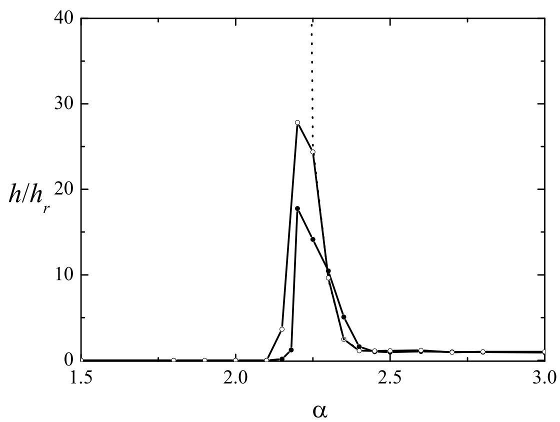

Fig. 5. Normalized Hamming distance ratio, h/h_r, as a function of α, for $N = 100$ and $F = 50$ on a random network (full dots), and on the one-dimensional array, either excluding (empty dots) or including (dotted line) twisted states.

Cross synchronization occurs also on the one-dimensional array. In this case, however, its detection through the Hamming distance is more tricky due to the presence of twisted states below the transition. In realizations where some of the phases are synchronized while other are distributed in twisted states the clustering structures are maximally different and, consequently, h attains very large values. The dotted line in Fig. 5 represents the ratio h/h_r as a function of α for realizations on the one-dimensional array. To compare with random networks, thus, we exclude from the calculation of h the realizations where twisted states are found. Empty dots in Fig. 5 stand for the result of this procedure. The phenomenon of cross synchronization is qualitatively the same as, but stronger than, in random networks.

4 Conclusion

Our study of synchronization in a population of coupled multi-phase oscillators has been motivated by the affinity between the emergence of cultural domains in Axelrod's model [1,2,3] and the occurrence of coherent behaviour in distributed interacting systems [7,8,9]. The state of each oscillator is characterized by a set of phases, which represent the cultural features of Axelrod's model. We focused our attention on the long-time asymptotic organization of the oscillators into synchronized or unsynchronized configurations, under the effect of pair interactions which are attractive if the individual states are similar, but are absent if the difference between individual states grows beyond a given threshold.

Analytic calculations suggest that, as a function of the interaction threshold, the population undergoes a synchronization transition. If the threshold is too small, individual states are generally too different from each other, and the population fails to collectively synchronize. Conversely, a large threshold enables the attractive interactions and promotes synchronization. The same calculations show that the transition becomes sharper if the number of phases per oscillator grows. On the other hand, the transition width is independent on the population size.

These analytical predictions were confirmed by numerical realizations of our system, both on ordered arrays and on random networks. Numerical results also revealed that, regarding the organizational structure of the population, the most interesting regime occurs precisely around the transition. We first found that, in this region, the synchronized state undergoes disaggregation into clusters of mutually synchronized oscillators, as observed in other systems of interacting oscillators [9]. The number of clusters and, consequently, their size, varies with the interaction threshold.

Remarkably, moreover, the segregation of oscillators into clusters is not necessarily associated with the synchronization of all the individual phases. In particular, two oscillators may belong to the same synchronized cluster with respect to some of their phases –which, in the asymptotic state, adopt identical values– but to different clusters with respect to the other phases. This phenomenon, which we have called cross synchronization, discloses a highly complex form of collective organization, consisting of coherent clusters mixed with respect to the internal variables which specify the individual states. It is interesting to note that Axelrod's model does not exhibit a regime corresponding to cross synchronization. This kind of regime would however make sense as a possible distribution of culture diversity over a real population, in the form of partially overlapping cultural domains. We have here presented a preliminary quantitative analysis of cross synchronization –aiming, mainly, at giving a means to detect the phenomenon– but, clearly, much further work is necessary to fully characterize its statistical features.

Acknowledgements

We acknowledge financial support from CONICET (PIP 5114), ANPCyT (PICT 04/943), and SECTyP (UNCuyo), Argentina.

References

1. R. Axelrod, J. Conflict Resolution **41**, 203 (1997). Reprinted in R. Axelrod, *The Complexity of Cooperation* (Princeton University Press, Princeton, 1997).
2. C. Castellano, S. Fortunato, and V. Loreto, arXiv:0710.3256v1 [physics.soc-ph].
3. C. Castellano, M. Marsili, and A. Vespignani, Phys. Rev. Lett. **85**, 3536 (2000).
4. K. Klemm, V. M. Eguíluz, R. Toral, and M. San Miguel, Phys. Rev. E **67**, 045101R (2003).
5. K. Klemm, V. M. Eguíluz, R. Toral, and M. San Miguel, Phys. Rev. E **67**, 026120 (2003).
6. F. Vázquez and S. Redner, EPL **78**, 18002 (2007).

7. G. B. Ermentrout and N. Kopel, SIAM J. Math. Anal. **15**, 215 (1984).
8. H. Sakaguchi, S. Shinimoto, and Y. Kuramoto, Prog. Theor. Phys. **77**, 1005 (1987).
9. S. C. Manrubia, A. S. Mikhailov, and D. H. Zanette, *Emergence of Dynamical Order. Synchronization Phenomena in Complex Systems* (World Scientific, Singapore, 2004).
10. Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
11. M. Newman, A.-L. Barabasi, and D. J. Watts, eds., *The Structure and Dynamics of Networks* (Princeton University Press, Princeton, 2006).
12. D. A. Wiley, S. H. Strogatz, and M. Girvan, Chaos **16**, 015103 (2006).